Journal of Statistical Physics, Vol. 33, No. 2, 1983

Some Remarks about Continuity Properties of Local Maxwellians and an Existence Theorem for the BGK Model of the Boltzmann Equation[1]

Jacek Polewczak[2] and William Greenberg[3]

Received September 16, 1982; revised March 1, 1983 and June 7, 1983

Continuity of local Maxwellians in various topologies of L^1 is studied. The existence and convergence of approximate solutions of the nonlinear BGK model of the Boltzmann equation are proved.

KEY WORDS: BGK equation; Boltzmann equation; local Maxwellian.

1. INTRODUCTION

Difficulties in dealing with the Boltzmann collision operator have led to a variety of models of the Boltzmann equation. In many of these, the original collision term is replaced with a simpler one which is treatable by known mathematical tools, but which still reflects certain characteristic features of the original collision operator, for example, the conservation laws (mass, momentum, energy) and the Boltzmann inequality.

The creators of the Bhatnagar–Gross–Krook (BGK) model[(1,2,3)] were less concerned with the simplicity of the mathematical properties of their model; instead they replaced (speaking from a physical point of view) "a large amount of detail of the two-body interaction which is contained in the collision term" by a model "which retains only the qualitative and average properties of the true operator" (see Ref. 1). The BGK model physically

[1] This work was supported in part by Department of Energy Grant No. DE-AS05-80ER10711.

[2] Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803.

[3] Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061.

0022-4715/83/1100-0307$03.00/0

describes a gas strongly tending to a Maxwellian distribution. This "tendency" is reflected also in the fact that, for the spatially homogeneous gas, we have exact solutions.

Let $f: \Omega \times \mathbb{R}^3 \to \mathbb{R}^+ \cup \{0\}$, where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, $x \in \Omega$ is the spatial variable, $\xi \in \mathbb{R}^3$ the velocity variable, and $f(x,\xi)$ represents the density distribution at the point x with velocity ξ. Denote by D_0 the set $D_0 = \{f \in \mathsf{L}^1(\Omega \times \mathbb{R}^3): f \geqslant 0,\ (1+\xi^2)f \in \mathsf{L}^1(\Omega \times \mathbb{R}^3)\}$, and let $\rho(x) = \int_{\mathbb{R}^3} f(x,\xi)\,d\xi$ be the macroscopic density, $v(x) = \int_{\mathbb{R}^3} \xi f(x,\xi)\,d\xi/\rho(x)$ the macroscopic velocity, $e(x) = \int_{\mathbb{R}^3}[\xi - v(x)]^2 f(x,\xi)\,d\xi/2\rho(x)$ the macroscopic kinetic energy, and $T(x) = 2e(x)/3R$ the macroscopic temperature, where R is the Boltzmann constant. Define pointwise the operator

$$\mathsf{P}(f)(x,\xi) = \frac{\rho(x)}{\left[2\pi RT(x)\right]^{3/2}} \exp \frac{-\left[\xi - v(x)\right]^2}{2RT(x)}$$

The BGK collision operator and the BGK equation can be written as

$$J(f)(x,\xi) = \nu\left[\mathsf{P}(f)(x,\xi) - f(x,\xi)\right]$$

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla_x f = J(f), \qquad t > 0 \tag{1.1}$$

$$f(0,x,\xi) = f_0(x,\xi)$$

Although the collision frequency ν can be a function of ξ, x, t or even f itself, we shall assume the ν is a positive constant.

Conservation of mass, momentum, and energy is expressed by the equalities

$$\int_{\mathbb{R}^3} \psi_i J(f)\,d\xi = 0, \qquad \text{a.e. in } x \tag{1.2}$$

for $i = 0,1,2,3,4$ and $f \in D_0$, where ψ_i are the collision invariants $\psi_0 = 1$, $\psi_i = \xi_i$ for $i = 1,2,3$ (components of ξ), and $\psi_4 = \xi^2$. Formally, at least, we have also the Boltzmann inequality

$$\int_{\mathbb{R}^3} J(f) \ln f\,d\xi \leqslant 0 \tag{1.3}$$

for fixed x, with equality if and only if $f = \mathsf{P}(f)$. (1.2) and (1.3) represent characteristic features of kinetic models of Boltzmann operators.

If we consider the spatially homogeneous BGK equation [$f(x,\xi)$ independent of x], then (1.2) implies that the zeroth, first, and second moments of a solution of (1.1) are constant in time. This suggests that the following set should be of some interest: $D_1 = \{f \in L^1(\mathbb{R}^3): f \geqslant 0,\ \int_{\mathbb{R}^3} \psi_i f\,d\xi = M_i,\ i = 0,1,2,3,4\}$ for fixed M_i. Obviously, $\mathsf{P}: D_1 \to D_1$, even more, it is a constant operator on D_1. Hence considered on D_1 (1.1) becomes a linear inhomogeneous equation:

$$\frac{\partial f}{\partial t} = \nu[\mathsf{P} - f], \qquad f(0,\xi) = f_0(\xi)$$

where $\mathsf{P}(f) = \mathsf{P}$ for $f \in D_1$. Therefore we may write the solution of the spatially homogeneous equation as

$$f(t,\xi) = e^{-\nu t} f_0(\xi) + (1 - e^{-\nu t})\mathsf{P}(f_0)(\xi) \tag{1.4}$$

and we see that f tends to a Maxwellian distribution exponentially when $t \to \infty$.

Define $D_2 \subset \mathsf{L}^1(\mathbb{R}^3)$ by $D_2 = \{f \in \mathsf{L}^1(\mathbb{R}^3) : f \geqslant 0,\ \int_{\mathbb{R}^3} \psi_i f\, d\xi = M_i,\ i = 0, 1, 2, 3,\ \int_{\mathbb{R}^3} \psi_4 f\, d\xi \leqslant M_4\}$. We complete this introduction with the following lemma by Gibbs:

Lemma 1 (Gibbs)[4,5]. Let $p \in D_2$ be the uniquely determined Maxwellian satisfying $\int_{\mathbb{R}^3} \xi^2 p\, d\xi = M_4$. Then for each $f \in D_2$,

$$\int_{\mathbb{R}^3} p \ln p\, d\xi \leqslant \int_{\mathbb{R}^3} f \ln f\, d\xi$$

and equality holds only if $f = p$.

2. SEMILINEAR EVOLUTION EQUATIONS

Despite the premier role the BGK equation has played in rarefied gas dynamics, there is at present no existence proof even for local solutions of the mild initial value problem. In the next section we will study properties of the BGK collision operator which will clarify the difficulty in obtaining a local existence theorem, and will, as well, prove the existence of a "generalized" solution. In order to motivate this study of continuity properties of local Maxwellians, we summarize, in this section, the existence theory for semilinear differential equations.

For X an arbitrary Banach space, U the generator of a strongly continuous linear semigroup $t \to T(t)$, $F : [0,T] \times D \to X$ a given (nonlinear) function of $t \in [0,T] \subset \mathbb{R}_+$ and $x_0 \in D \subset X$, a strong solution of the semilinear evolution equation

$$\frac{\partial x}{\partial t} = Ux + F(t,x), \qquad x(0) = x_0 \in D \tag{2.1}$$

is an absolutely continuous function $t \to x(t) \in D$, differentiable a.e. with $x(t) \in \mathsf{D}(U)$ a.e., and satisfying (2.1). If F is continuous in an appropriate topology (and the integral is defined correspondingly), then a continuous function $x(t) \in D$ which satisfies

$$x(t) = T(t)x_0 + \int_0^t T(t-s)F[s, x(s)]\,ds \tag{2.2}$$

is said to be a mild solution of the initial value problem.

In the situation where F is continuous in the original Banach space topology, a number of existence criteria are known:

Theorem 1 (Refs. 6 and 7, pp. 335–339). Assume $D \subset X$ is locally closed, F is jointly continuous, and

$$\lim_{h\to 0^+} (1/h)\operatorname{dist}\left[T(h)x + \int_t^{t+h} T(t+h-s)F(t,x)\,ds; D\right] = 0 \tag{2.3}$$

for all $x \in D$ and $t \in [0, T]$. Then there exists a mild local solution if any of the following is true:

(i) $T(t)$ compact for all $t > 0$,

(ii) $\operatorname{Ran} F$ compact,

(iii) F (locally) dissipative.

In the event that F fails to have the necessary topological properties in the original topology, it may be possible to obtain results in the weak topology. A general existence theorem related to weak topologies of Banach spaces is the following.

Theorem 2 (Polewczak)[8]. Assume $D \subset X$ weakly closed, F jointly weakly sequentially continuous, and (2.3) valid uniformly on strongly compact sets of D. Then there exists a mild local solution if either

(ia) $T(\cdot)$ jointly weakly sequentially continuous,

(ib) $\operatorname{Ran} F$ weakly relatively compact,

or

(iia) $T(t)$ weakly compact and the map $t \to T(t)$ is weakly continuous, uniformly on the unit sphere of X, for $t > 0$,

(iib) for all bounded $G \subset D$ and $x \in G$, $\|F(t,x)\| \leqslant M_G(t)$ for some locally integrable M_G,

and the integral is taken in the Bochner sense.

The previous theorem is the sort of result one would hope to apply to the BGK equation. Indeed, as we shall see, a natural domain of definition for the nonlinear BGK operator is a weakly compact set D in a certain Banach space X. Moreover, D is an invariant domain for the semilinear evolution equation (2.1).

For $F = F(x)$ autonomous and D weakly compact, an easy modification of the theorem can be obtained. However, we first define the notion of an approximate solution [for F autonomous and $T(t)$ a contraction semigroup].

Definition 1. An approximate solution of (2.2) is a sequence $\{f_n(\cdot)\}_{n=1}^{\infty}$ of functions $f_n : [0, T] \to D, T > 0$, and a sequence $\{\{t_i^n\}\}_{n=1}^{\infty}$ of partitions $\{t_i^n\}_{i=1}^{N(n)}$ of $[0, T]$ such that $t_{i+1}^n - t_i^n \leqslant \epsilon_n \to 0$, and

(a) $$f_n(0) = x_0, \qquad f_n(t_i^n) \in D$$

(b) $$f_n(t) = T(t - t_i^n) f_n(t_i^n) + \int_{t_i^n}^{t} T(t - s) F(f_n(t_i^n))\, ds,$$

$$\text{for} \quad t \in [t_i^n, t_{i+1}^n)$$

(c) $$\| f_n(t_{i+1}^n) - f_n(t_i^n) \| \leqslant \epsilon_n (t_{i+1}^n - t_i^n)$$

(d) $$\left\| f_n(t) - T(t) x_0 - \int_0^t T(t - s) F[f_n(\gamma_n(s))]\, ds \right\| \leqslant t_i^n \epsilon_n$$

for $t \in [t_i^n, t_{i+1}^n)$ where $\gamma_n(t) = t_i^n$ if $t \in [t_i^n, t_{i+1}^n)$ and $\gamma_n(T) = T$.

Let us note that from (b) one sees that for each n, f_n is the mild solution on $[t_i^n, t_i^{n+1})$ to the nonhomogeneous linear equation $\dot{x}(t) = Ux(t) + F[f_n(t_i^n)]$, $x(t_i^n) = f_n(t_i^n)$. More detailed analysis of an approximate solution can be found in Ref. 7 (pp. 322–330).

Corollary 1. Let X be a Banach space, U the generator of a strongly continuous semigroup $T(t)$, D a closed set, and F locally bounded. Then for each $x_0 \in D$, (2.2) has an approximate solution on $[0, T]$ for some $T > 0$ if

(i) (2.3) is valid uniformly on strongly compact sets of D. If (2.2) has an approximate solution on $[0, T]$ for some $T > 0$, and

(ii) D is weakly compact, $T(t)D \subseteq D$ for $t \geqslant 0$ and $F = \sum_{i=1}^{m} \alpha_i F_i$ with $\alpha_i \in \mathbb{R}$ and $F_i : D \to D$,

then the approximate solution contains a subsequence $\{f_{n_i}\}_{i=1}^{\infty}$ which converges weakly and uniformly (on $[0, T]$) to a limit $f(t)$ and $f : [0, T] \to D$ is weakly continuous. Finally, if in addition to condition (ii),

(iii) $F : D \to X$ is weakly sequentially continuous,

then $f(t)$ is a mild (global) solution to (2.2). The integral in (2.2) is taken in the Bochner sense.

Let us point out that both (ii) of Corollary 1 and condition (ia) of Theorem 2 are essential in proving equicontinuity of the family of approximate solutions. Here, however, we do not require joint weak sequential continuity of $T(\cdot)$; instead $T(t)$ is assumed to be D invariant for $t \geqslant 0$. This together with the weak compactness of D gives the necessary continuity property of $T(\cdot)$ on $(0, T] \times D$ (see Ref. 8, Theorem II.0). However, since $F : D \to X$ (not into D), $T(t)F(x)$ may not be weakly continuous in $t > 0$, uniformly for $x \in D$. We can overcome this difficulty by imposing an additional assumption on F as is done in (ii) of the Corollary. Indeed,

$T(t)F(x) = \sum_{i=1}^{m} \alpha_i T(t)F_i(x)$ together with the joint weak sequential continuity of $T(\cdot)$ on $(0, T] \times D$ gives the weak continuity of $T(t)F(x)$ in $t > 0$, uniformly for $x \in D$. From this equicontinuity of the family of approximate solutions is apparent.

3. CONTINUITY PROPERTIES OF LOCAL MAXWELLIANS AND EXISTENCE OF GENERALIZED SOLUTIONS

Let us define the Banach spaces $\mathsf{L}^1_{1+\xi^2}(\mathbb{R}^3) = \{f \in \mathsf{L}^1(\mathbb{R}^3) : \|f\|_2 \equiv \int_{\mathbb{R}^3}(1 + \xi^2)|f|\, d\xi < \infty\}$ and $\mathsf{L}^1_{1+\xi^2}(\Omega \times \mathbb{R}^3) = \{f \in \mathsf{L}^1(\Omega \times \mathbb{R}^3) : \|f\|_2 \equiv \iint_{\Omega \times \mathbb{R}^3}(1 + \xi^2)|f|\, d\xi\, dx < \infty\}$. Since $\mathsf{L}^1_{1+\xi^2}(\mathbb{R}^3) \subset \mathsf{L}^1(\mathbb{R}^3)$, we shall refer to the L^1 topology and the $\mathsf{L}^1_{1+\xi^2}$ topology in a natural way, and likewise for $\mathsf{L}^1_{1+\xi^2}(\Omega \times \mathbb{R}^3)$.

We define the set D by $D = \{f \in \mathsf{L}^1(\Omega \times \mathbb{R}^3) : f \geqslant 0,\ \iint_{\Omega \times \mathbb{R}^3} f\, d\xi\, dx = M_0,\ \iint_{\Omega \times \mathbb{R}^3} \psi_4 f\, d\xi\, dx \leqslant M_4,\ H(f) \leqslant M_5\}$, where $M_0, M_4 > 0$, and the functional $H : D \to \mathbb{R}$ by $H(f) = \iint_{\Omega \times \mathbb{R}^3} f \ln f\, d\xi\, dx$.

The set D is a natural domain on which to define the BGK collision operator by virtue of the conservation laws and the H theorem. Indeed, finiteness of mass, momenta, and energy are necessary to define P. Further, D is an invariant domain for the semilinear evolution equation. In fact, as we shall see later, (2.3) is satisfied for all $x \in D$.

It becomes immediately evident, however, that serious problems arise in proving an existence theorem in the original L^1 topology. For although D is L^1 closed, it is not L^1 compact, and, even more, P is not L^1 continuous (original topology). The closedness of D follows from the fact that the functional $H : D \to \mathbb{R}$ is lower semicontinuous in L^1 (see Ref. 9, Proposition 5.1). Regarding L^1 compactness of D, it is not difficult to construct a family of functions in D which is not compact in L^1. To see that P is not continuous, it is enough to consider $f_n \to f$ in L^1 such that $\{\int_{\mathbb{R}^3} \xi^2 f_n\, d\xi\}_{n=1}^{\infty}$ is not subsequence convergent pointwise a.e. in x, whence $\mathsf{P}(f_n)$ cannot converge in L^1.

The topological behavior of P is improved in $\mathsf{L}^1_{1+\xi^2}$, but the compactness problem of D still remains. More precisely, D is $\mathsf{L}^1_{1+\xi^2}$ closed and $\mathsf{P} : D \to D$ is $\mathsf{L}^1_{1+\xi^2}$ continuous, but D is not $\mathsf{L}^1_{1+\xi^2}$ compact. The properties of D are obvious. To see the continuity of P, let $f_n \in D$ and $f_n \to f$ in $\mathsf{L}^1_{1+\xi^2}$. Then, $f_n \to f$ in L^1, $\|\mathsf{P}(f_n)\|_2 \to \|\mathsf{P}(f)\|_2$ via the conservation laws, and $\int_{\mathbb{R}^3}(1 + \xi^2) f_n\, d\xi \to \int_{\mathbb{R}^3}(1 + \xi^2) f\, d\xi$ in $\mathsf{L}^1(\Omega)$. Therefore there exists a subsequence $\{f_{n_i}\}$ such that $\int_{\mathbb{R}^3}(1 + \xi^2) f_{n_i}\, d\xi$ converges a.e. in x to $\int_{\mathbb{R}^3}(1 + \xi^2) f\, d\xi$, and (after possibly passing to another subsequence) $\int_{\mathbb{R}^3} \psi_j f_{n_i}\, d\xi \to \int_{\mathbb{R}^3} \psi_j f\, d\xi$ a.e. in x, $j = 0, 1, 2, 3, 4$. As a result, we have $\mathsf{P}(f_{n_i}) \to \mathsf{P}(f)$ a.e. in x and ξ. This gives $\mathsf{P}(f_n) \to \mathsf{P}(f)$ in measure on every subset of finite measure, and from the convergence of $\|\mathsf{P}(f_n)\|_2$, $\mathsf{P}(f_n) \to \mathsf{P}(f)$ in $\mathsf{L}^1_{1+\xi^2}$.

This singular behavior in the strong topologies suggests that the BGK equation might better be studied in the weak topologies. We begin the study of weak continuity properties with some simple, but important, generalizations of Gibbs' lemma to the inhomogeneous case. This will lead to sufficient conditions for the weak continuity of P as a map on the domain $D \subset \mathsf{L}^1(\Omega \times \mathbb{R}^3)$.

Lemma 1 (See Ref. 9, Proposition 5.1). D is a convex closed set in L^1, and for each $f \in D$, $f \ln f \in \mathsf{L}^1(\Omega \times \mathbb{R}^3)$. Thus, D is weakly closed, and the functional $H: D \to \mathbb{R}$ is weakly lower semicontinuous.

Note that weak lower semicontinuity follows from the (strong) lower semicontinuity of the convex functional H (see Ref. 10, p. 11, Corollary 2.2). Since, for x fixed, $\mathsf{P}(f)(x, \cdot)$ is a Maxwellian, we have also the following:

Lemma 2. For each $f \in D$, $H(\mathsf{P}(f)) \leqslant H(f)$. Hence $\mathsf{P}: D \to D$.

Lemma 3. For fixed $f \in D$ suppose $g \in D$ satisfies

$$\int_{\mathbb{R}^3} \psi_i f \, d\xi = \int_{\mathbb{R}^3} \psi_i g \, d\xi \qquad \text{a.e. in } x$$

for $i = 0, 1, 2, 3, 4$, and

$$H(\mathsf{P}(f)) = H(g)$$

Then $g = \mathsf{P}(f)$ in $\mathsf{L}^1(\Omega \times \mathbb{R}^3)$.

Proof. Note first that if $\{f_i\}_{i=1}^n \subset D$, $\alpha_i \geqslant 0$, $\sum_{i=1}^n \alpha_i = 1$, then

$$\mathsf{P}\left(\sum_{i=1}^{n} \alpha_i f_i \right) = \mathsf{P}\left[\sum_{i=1}^{n} \alpha_i \mathsf{P}(f_i) \right]$$

and in particular, P is idempotent. For P depends upon f only through its first three moments.

Assume that $g \neq \mathsf{P}(f)$ in $\mathsf{L}^1(\Omega \times \mathbb{R}^3)$. Then

$$H(\mathsf{P}(f)) = H\big(\mathsf{P}(\tfrac{1}{2} g + \tfrac{1}{2}\mathsf{P}(f))\big) \leqslant H(\tfrac{1}{2} g + \tfrac{1}{2}\mathsf{P}(f)) < H(\mathsf{P}(f))$$

where the strong convexity of $x \ln x$, $x \geqslant 0$, has been used in the last inequality, thus obtaining a contradiction. ∎

Lemma 4 (Refs. 5 and 11). D is a weakly compact set of $\mathsf{L}^1(\Omega \times \mathbb{R}^3)$. Even more, for each sequence $\{f_n\} \subset D$ there exists a subsequence $\{f_{n_i}\}$ and an f such that $\iint_{\Omega \times \mathbb{R}^3} f_{n_i} g \, d\xi \, dx \to \iint_{\Omega \times \mathbb{R}^3} f g \, d\xi \, dx$ for all measurable g such that $|g(x, \xi)| \leqslant c(1 + \xi^2)^k$, $k < 1$.

The following question arises. Is P L^1-weakly continuous as a map on D? We shall show that the answer is no. However, it is instructive first to study the implications of (weak) continuity. To that end, let us note that D

is weakly metrizable since L^1 is separable and D is weakly compact (see Ref. 13, Theorem 3, p. 434), and thus weak continuity is equivalent to weak sequential continuity, and hence to weak sequential closedness. We have the following.

Theorem 1. Suppose $\{f_n\} \subset D$ and $f_n \to f$, $\mathsf{P}(f_n) \to g$ weakly in $\mathsf{L}^1(\Omega \times \mathbb{R}^3)$. Then $g = \mathsf{P}(f)$ if and only if

$$\text{(i)} \qquad \int_{\mathbb{R}^3} \xi^2 f \, d\xi = \int_{\mathbb{R}^3} \xi^2 g \, d\xi \qquad \text{a.e. in } x$$

and

$$\text{(ii)} \qquad H(g) \leqslant H(\mathsf{P}(f))$$

Proof. Since $f_n \to f$ weakly in $\mathsf{L}^1(\Omega \times \mathbb{R}^3)$, $\int_\Omega h(x)(\int_{\mathbb{R}^3} \mathsf{P}(f_n)\, d\xi)\, dx \to \int_\Omega h(x)(\int_{\mathbb{R}^3} f\, d\xi)\, dx$ for each $h \in \mathsf{L}^\infty(\Omega)$. On the other hand,

$$\int_\Omega h(x)\left(\int_{\mathbb{R}^3} \mathsf{P}(f_n)\, d\xi\right) dx \to \int_\Omega h(x)\left(\int_{\mathbb{R}^3} g\, d\xi\right) dx$$

and thus

$$\int_{\mathbb{R}^3} f\, d\xi = \int_{\mathbb{R}^3} g\, d\xi \qquad \text{a.e. in } x$$

Using Lemma 4 and similar arguments, we have

$$\int_{\mathbb{R}^3} \psi_i f\, d\xi = \int_{\mathbb{R}^3} \psi_i g\, d\xi \qquad \text{a.e. in } x$$

for $i = 0, 1, 2, 3$, and for $i = 4$ by assumption (i). Hence $\mathsf{P}(g) = \mathsf{P}(f)$. Using Lemma 2, $H(\mathsf{P}(f)) \leqslant H(g)$ and thus, by (ii), $H(\mathsf{P}(f)) = H(g)$. Then Lemma 3 gives $\mathsf{P}(f) = g$. The converse is trivial. ■

Remark 1. Assumption (i) would be satisfied if boundedness of one of the higher moments of $\{f_n\}$ is assumed; i.e., $\int\int_{\Omega \times \mathbb{R}^3} |\xi|^k f_n \, d\xi\, dx \leqslant c$ for $k > 2$ and for all n. If (i) holds, then (ii) is equivalent to $H(\mathsf{P}(f)) = H(g)$. In the spatially homogeneous case, (ii) is always satisfied if (i) is valid.

To characterize the weak continuity of P, it is enough to consider the spatially homogeneous case. Thus, let D_h be defined by

$$D_h = \left\{ f \in \mathsf{L}^1(\mathbb{R}^3) : f \geqslant 0, \int_{\mathbb{R}^3} f\, d\xi = M_0, \int_{\mathbb{R}^3} \xi^2 f\, d\xi \leqslant M_4, \right.$$
$$\left. \int_{\mathbb{R}^3} f \ln f\, d\xi \leqslant M_5 \right\}, \qquad \text{where} \quad M_0, M_4 > 0$$

Theorem 2. $\mathsf{P} : D_h \to D_h$ is not weakly continuous.

Proof. Suppose $\{f_n\}_{n=1}^{\infty} \subset D_h$, $f_n \to f$ in $\mathsf{L}^1(\mathbb{R}^3)$ and assume $\mathsf{P}: D_h \to D_h$ is L^1 weakly continuous. An easy calculation shows that

$$\sup_n \int_{\mathbb{R}^3} \xi^4 \mathsf{P}(f_n)\, d\xi < \infty.$$

By an argument analogous to Lemma 4 (after possibly passing to a subsequence), $\mathsf{P}(f_{n_i}) \to g$ weakly in $\mathsf{L}^1(\mathbb{R}^3)$ and $\int_{\mathbb{R}^3} \xi^2 \mathsf{P}(f_{n_i})\, d\xi = \int_{\mathbb{R}^3} \xi^2 f_{n_i}\, d\xi \to \int \xi^2 g\, d\xi$. Since P is weakly continuous, $\mathsf{P}(f) = g$ and $\int_{\mathbb{R}^3} \xi^2 f_{n_i}\, d\xi \to \int_{\mathbb{R}^3} \xi^2 f\, d\xi$. This proves the $\mathsf{L}^1(\mathbb{R}^3)$ topology of D_h is equivalent to the $\mathsf{L}^1_{1+\xi^2}(\mathbb{R}^3)$ topology of D_h, which is clearly false. ∎

Let us consider $X = \mathsf{L}^1(\Omega \times \mathbb{R}^3)$, $Y = \mathsf{L}^1_{1+\xi^2}(\Omega \times \mathbb{R}^3)$ and D as defined at the start of this section, $Uf = -\xi(\partial f / \partial x)$ together with appropriate differentiability and boundary conditions on Ω so that A generates a strongly continuous semigroup on X satisfying $T(t)D \subset D$ (see, for example, Ref. 9), and moreover, the restriction of $T(t)$ to Y [also denoted by $T(t)$] exists and is a strongly continuous semigroup in Y. Now let us define $F(f) = \nu[\mathsf{P}(f) - f]$ for f in D. By virtue of Theorem 2, (iii) of Corollary 1 in Section 2 cannot be satisfied. However, the lemmas above do prove existence of an approximate solution in $Y \subset X$ and its weak convergence in X. Indeed, since F is continuous in Y, uniform convergence of (2.3) on strongly compact sets of $D \subset Y$ is equivalent to the uniform convergence of

$$\lim_{h \to 0^+} 1/h \operatorname{dist}(T(h)f + hF(f); D) = 0 \tag{3.1}$$

on strongly compact sets of $D \subset Y$, where $\operatorname{dist}(f; D)$ is the distance function from f to D in the Banach space Y. Furthermore, because of the strong continuity of $T(t)$ and F in Y and $T(t)D \subseteq D$ for $t \geqslant 0$, (3.1) is satisfied uniformly on compact sets of $D \subset Y$ if

$$\lim_{h \to 0^+} 1/h \operatorname{dist}(f + hF(f); D) = 0 \tag{3.2}$$

uniformly on compact sets of $D \subset Y$. Finally, $f + hF(f) = (1 - h\nu)f + h\nu\mathsf{P}(f) \in D$ for $0 \leqslant h \leqslant \nu^{-1}$ and for all f in D, so we see that (3.2) converges uniformly in D. Since D is closed in Y, we have the following.

Theorem 3. The BGK equation (1.1) with $x_0 \in D$ has an approximate solution in $\mathsf{L}^1_{1+\xi^2}(\Omega \times \mathbb{R}^3)$, where the integrals are Riemann integrals in $\mathsf{L}^1_{1+\xi^2}(\Omega \times \mathbb{R}^3)$.

Theorem 3 together with the continuity of the imbedding $Y \subset X$ gives the existence of an approximate solution in X. Since D is weakly compact in X, assumption (ii) of Corollary 1 in Section 2 is satisfied. We have the following.

Corollary 1. The approximate solution contains a subsequence $\{f_{n_i}\}_{i=1}^{\infty}$ which converges weakly in $L^1(\Omega \times \mathbb{R}^3)$ and uniformly (on $[0, T]$) to a limit $\lim_{i\to\infty} f_{n_i}(t) = f(t)$, and $f: [0, T] \to D$ is weakly continuous.

We remark that, owing to the lack of a weak continuity property of F, we cannot conclude that $f(t)$ is a (mild) solution of (1.1). Such a limiting function $f(t)$ may be considered a "generalized" solution of the BGK equation. This is similar to the situation in Ref. 12, where the authors were likewise unable to say in what sense the limit of their approximate solutions to the Boltzmann equation satisfied the original equation.

REFERENCES

1. C. Cercignani, *Theory and Application of the Boltzmann Equation* (Elsevier, New York, 1975).
2. P. L. Bhatnagar, E. P. Gross, and M. Krook, *Phys. Rev.* **94**:511 (1954).
3. P. Welander, *Ark. Fys.* **7**:507 (1954).
4. J. W. Gibbs, *Collected Works*, Vol. II (New Haven, 1906), p. 130.
5. D. Morgenstern, *J. Rational Mech. Anal.* **4**:533 (1955).
6. N. Pavel, *Nonlin. Anal. Theor. Meth. Appl.* **1**:187 (1976).
7. R. H. Martin, Jr., *Nonlinear Operators and Differential Equations in Banach Spaces* (Wiley, New York, 1976).
8. J. Polewczak, Semilinear evolution equations in weak topologies of non-reflexive Banach spaces, preprint, 1981.
9. J. Voigt, The H-theorem for Boltzmann type equations, preprint, 1979. Some interesting aspects of the preprint do not appear in the published version in *J. Reine Angew. Math.* **326**:198 (1981).
10. I. Ekeland and R. Temam, *Convex Analysis and Variational Problems* (North-Holland, New York, 1976).
11. L. Arkeryd, *Arch. Rational Mech. Anal.* **45**:1 (1972).
12. W. Greenberg, J. Voigt, P. F. Zweifel, *J. Stat. Phys.* **21**:649 (1979).
13. N. Dunford and J. T. Schwartz, *Linear Operators*, Part I (Interscience, New York, 1958).